YACC: **Y**et **A**nother **C**hurch **C**alculus

joint work by

Franco Barbanera Mariangiola Dezani Ugo de'Liguoro Betti Venneri

ITRS 2024 - Tallinn, July the 9th 2024

Plan of the talk

Simple Types

$$T ::= \varphi \mid T \rightarrow T$$

Simple Types

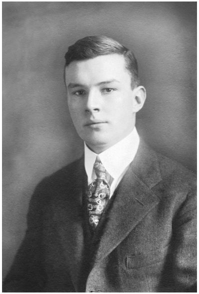
$$T ::= \varphi \mid T \rightarrow T$$

à la Curry

Simple Types

$$T ::= \varphi \mid T \to T$$

à la Curry



*FUNCTIONALITY IN COMBINATORY LOGIC**

BY H. B. CURRY

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE COLLEGE

Communicated September 20, 1934

1. *Introduction.*—In an attempt to resolve the foundations of logic and mathematics into their elements, it has occurred to several persons that certain notions, ordinarily taken as primitive, could be analyzed into constituents of much simpler nature. Among such notions are, on the one hand, various processes of substitution, and the use of variables generally; and, on the other hand, the categories of logic—such as proposition, propositional function and the like—together with the intuitions by which we tell what entities belong to them.

For a theory concerned with an analysis of these notions I have proposed the name *combinatory logic* (*Amer. Jour. Math.*, **52,** 511 (1930)). This is

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$$\frac{}{\Gamma, x : T \vdash_{Cu} x : T}\,[Cu\textsc{Var}] \qquad \frac{\Gamma, x : T \vdash_{Cu} P : U}{\Gamma \vdash_{Cu} \lambda x.P : T \rightarrow U}\,[Cu \rightarrow I]$$

$$\frac{\Gamma \vdash_{Cu} P : T \rightarrow U \quad \Gamma \vdash_{Cu} Q : T}{\Gamma \vdash_{Cu} PQ : U}\,[Cu \rightarrow E]$$

Simple Types

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à la Church

Simple Types

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THE JOURNAL OF SYMBOLIC LOGIC

The Journal of Symbolic Logic

Article contents

Extract

References

A formulation of the simple theory of types

Published online by Cambridge University Press: 12 March 2014

Alonzo Church Show author details

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Extract

The purpose of the present paper is to give a formulation of the simple theory of types which incorporates certain features of the calculus of λ-conversion. A complete incorporation of the calculus of λ-conversion into the theory of types is impossible if we require that λ*x* and juxtaposition shall retain their respective meanings as an abstraction operator and as denoting the application of function to argument. But the present partial

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$$\frac{}{\Gamma, x : T \vdash_{Ch} x : T}\ [Ch\textsc{Var}] \qquad \frac{\Gamma, x : T \vdash_{Ch} P : U}{\Gamma \vdash_{Ch} \lambda x : T.P : T \to U}\ [Ch \to I]$$

$$\frac{\Gamma \vdash_{Ch} P : T \to U \quad \Gamma \vdash_{Ch} Q : T}{\Gamma \vdash_{Ch} PQ : U}\ [Ch \to E]$$

Simple Types

$[Cu \to I]$

$$\frac{\Gamma, x : T \vdash_{Cu} P : U}{\Gamma \vdash_{Cu} \lambda x.P : T \to U}$$

$[Ch \to I]$

$$\frac{\Gamma, x : T \vdash_{Ch} P : U}{\Gamma \vdash_{Ch} \lambda x : T.P : T \to U}$$

Simple Types

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Intersection Types

$$\mu ::= \varphi \mid \omega \mid \mu \rightarrow \mu \mid \mu \cap \mu$$

Intersection Types

$$\mu ::= \varphi \mid \omega \mid \mu \to \mu \mid \mu \cap \mu$$

à la Curry

Notre Dame Journal of Formal Logic
Volume 21, Number 4, October 1980

An Extension of the Basic Functionality Theory for the λ-Calculus

M. COPPO and M. DEZANI-CIANCAGLINI

1 Introduction The first works about the assignment of types to terms of the λ-calculus (or combinatory logic) arose in the context of logical theories of types.* Church [2] presented a first-order system with types based on λ-conversion, and since then many systems of this kind have been proposed; a review of them is given in the introduction of [15]. The problem of adjoining new objects and deduction rules to combinatory logic is faced in a more general way in [6] and [7], since Curry is interested in studying the properties of

Intersection Types

$$\mu ::= \varphi \mid \omega \mid \mu \to \mu \mid \mu \cap \mu$$

à la Curry

$$\frac{}{\Gamma, x : \mu \vdash_\cap x : \mu}\,[\cap\textsc{Var}] \qquad \frac{}{\Gamma \vdash_\cap P : \omega}\,[\cap\omega]$$

$$\frac{\Gamma, x : \mu \vdash_\cap P : \nu}{\Gamma \vdash_\cap \lambda x.P : \mu \to \nu}\,[\cap \to I] \qquad \frac{\Gamma \vdash_\cap P : \mu \to \nu \quad \Gamma \vdash_\cap Q : \mu}{\Gamma \vdash_\cap PQ : \nu}\,[\cap \to E]$$

$$\frac{\Gamma \vdash_\cap P : \mu \quad \mu \leq \nu}{\Gamma \vdash_\cap P : \nu}\,[\cap \leq] \qquad \frac{\Gamma \vdash_\cap P : \mu \quad \Gamma \vdash_\cap P : \nu}{\Gamma \vdash_\cap P : \mu \cap \nu}\,[\cap I]$$

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$$\frac{}{\Gamma, x : \mu \vdash_\cap x : \mu}[\cap\text{VAR}] \qquad \frac{}{\Gamma \vdash_\cap P : \omega}[\cap\omega]$$

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à la Church

"what can we replace to ? in $\vdash \lambda x : ?.x : (\phi \to \phi) \cap (\psi \to \psi)$?"

Different roles of $\wedge$

1. a variable can have different types in different occurrences

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2. a variable can have different types in the same occurrence merged by an application of Rule [$\wedge$ I]

Different roles of ∧

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TIC: Typed Intersection Calculus

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$$\lambda x : (\phi \to \psi)\&\psi . x^{\phi \to \psi} x^{\phi} : (\phi \to \psi)\&\psi \to \psi$$

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3 type constructors for intersection

1. $\wedge$
    1.1 $x^{\phi \wedge \psi}$
    1.2 $\lambda x : \phi \wedge \psi$
    1.3 $\phi \wedge \psi \rightarrow \phi$
    1.4 $M : \phi \wedge \psi$

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    1.4 $M : \phi \wedge \psi$
2. $\&$
    2.1 $\lambda x : \phi \& \psi$
    2.2 $\phi \& \psi \to \phi$

3 type constructors for intersection

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   1.1 $x^{\phi \wedge \psi}$
   1.2 $\lambda x : \phi \wedge \psi$
   1.3 $\phi \wedge \psi \to \phi$
   1.4 $M : \phi \wedge \psi$
2. $\&$
   2.1 $\lambda x : \phi \& \psi$
   2.2 $\phi \& \psi \to \phi$
3. $\sqcap$
   3.1 $\lambda x : \phi \sqcap \psi$

TIC terms

$$M ::= x^{\sigma} \mid \lambda x : \kappa.M \mid MM \mid \Omega$$

TIC terms

$$M ::= x^\sigma \mid \lambda x : \kappa . M \mid MM \mid \Omega$$

$$\frac{}{\vdash x^\alpha : \alpha}\ [\textsc{Var}] \qquad\qquad \alpha ::= \varphi \mid \theta \to \alpha$$

TIC terms

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$$\frac{}{\vdash \Omega : \omega}\ [\omega]$$

$$\frac{\vdash M : \alpha \quad \iota(M, x) = \theta}{\vdash \lambda x : \theta.M : \theta \to \alpha}\ [\to I]$$

ι collects using & the types of x in M

TIC terms

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$$\frac{\vdash M : \omega \to \alpha}{\vdash M\Omega : \alpha}\,[\to E\omega]$$

$$\frac{\vdash M : \vartheta \to \alpha \quad \vdash N : \sigma \quad \vartheta \ltimes \sigma}{\vdash MN : \alpha}\,[\to E]$$

$\ltimes$ maps $\&$ in ϑ to $\wedge$ in σ

TIC terms

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$$\frac{\vdash M : \sigma \quad \vdash N : \tau}{\vdash M \bigwedge N : \sigma \wedge \tau}\ [\wedge I] \qquad (\lambda x : \kappa . M) \bigwedge (\lambda x : \kappa' . N) = \lambda x : \kappa \sqcap \kappa' . M \bigwedge N$$

TIC terms

$$M ::= x^{\sigma} \mid \lambda x : \kappa.M \mid MM \mid \Omega$$

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$$\frac{\vdash M : \sigma \quad \vdash N : \tau}{\vdash M \bigwedge N : \sigma \wedge \tau}\ [\wedge I]$$

Type Reconstruction

the function *der*(M) either returns a derivation with conclusion $\vdash M : \mathit{type}(M)$ or fails

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the function *der*(M) either returns a derivation with conclusion $\vdash M : type(M)$ or fails

$$der(MN) = \begin{cases} \dfrac{der(M)}{\vdash MN : \alpha}\,[\to E\omega] & \text{if } type(M) = \omega \to \alpha \text{ and } N = \Omega \\[2ex] \dfrac{der(M) \quad der(N) \quad \vartheta \ltimes \sigma}{\vdash MN : \tau}\,[\to E] & \text{if } type(M) = \vartheta \to \alpha \text{ and } type(N) = \sigma \text{ and } \vartheta \ltimes \sigma \\[2ex] \dfrac{der(\Pi_1(MN)) \quad der(\Pi_2(MN))}{\vdash MN : \tau_1 \wedge \tau_2}\,[\wedge I] & \text{if } type(\Pi_i(MN)) = \tau_i \text{ for } i = 1,2 \end{cases}$$

TIC reduction rules

$$[\beta\&] \qquad (\lambda x : \&_{i \in I}\sigma_i.M)N \longrightarrow M[x^{\sigma_i} := N_i]_{i \in I}$$

where $\sigma = type(N) \backsimeq \bigwedge_{i \in I}\sigma_i$ and $N_i = \widetilde{\Pi_i^{\sigma}}(N)$ for each $i \in I$

TIC reduction rules

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$[\beta\omega]$ $(\lambda x : \omega.M)\Omega \longrightarrow M$

TIC reduction rules

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$[\beta\omega]$ $\quad (\lambda x : \omega.M)\Omega \longrightarrow M$

$$[\beta\sqcap] \quad \frac{(\lambda x : \kappa_j.M_j)N_j \longrightarrow M'_j \quad j = 1, 2}{(\lambda x : \kappa_1 \sqcap \kappa_2.M_1 \bigwedge M_2)(N_1 \bigwedge N_2) \longrightarrow M'_1 \bigwedge M'_2}$$

Relations with $\vdash_\cap$

$\frown$replaces $\wedge$ and & with $\cap$

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$\widehat{\ }$ replaces $\wedge$ and $\&$ with $\cap$

$\| \ \|$ erases types and replaces Ω with $(\lambda x.xx)(\lambda x.xx)$

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Theorem

If $\vdash M : \zeta$, *then* $\{x : \widehat{\vartheta} \mid \iota(M, x) = \vartheta\} \vdash_\cap \|M\| : \widehat{\zeta}$.

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$\widehat{\ }$ replaces $\wedge$ and & with $\cap$

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Theorem

If $\vdash M : \zeta$, *then* $\{x : \widehat{\vartheta} \mid \iota(M,x) = \vartheta\} \vdash_\cap \|M\| : \widehat{\zeta}$.

Theorem

Given a principal typing $\langle \Gamma ; \mu \rangle$ *there are infinitely many TIC terms* M *such that* $\widehat{type(M)} = \mu$ *and* $\Gamma = \{x : \widehat{\vartheta} \mid \iota(M,x) = \vartheta\}$.

Main properties

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Theorem (Subject Reduction)

If $\vdash M : \tau$ *and* $M \longrightarrow N$*, then* $\vdash N : \tau$*.*

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Theorem

If $M \longrightarrow N$*, then* $\|M\| \longrightarrow_\beta \|N\|$*.*

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Theorem

If $M \longrightarrow N$, *then* $\|M\| \longrightarrow_\beta \|N\|$.

Theorem

If $\|M\| \longrightarrow_\beta P$, *then there is a TIC term* N *such that* $\|N\| = P$ *and either* $M \longrightarrow N$ *or* $N = M$.

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Theorem (Subject Reduction)

If $\vdash M : \tau$ and $M \longrightarrow N$, then $\vdash N : \tau$.

Theorem

If $M \longrightarrow N$, then $\|M\| \longrightarrow_\beta \|N\|$.

Theorem

If $\|M\| \longrightarrow_\beta P$, then there is a TIC term N such that $\|N\| = P$ and either $M \longrightarrow N$ or $N = M$.

Theorem (Characterisation of head normal forms)

A TIC term M has a head normal form iff $type(M) \neq \omega$.

Related works

Joe B. Wells and Christian Haack, Branching types, ESOP 2002:115–132

$$\Lambda(\text{join}\{i = \star, j = \star\}).\lambda x^{\{i=\phi, j=\psi\}}.x^{\{i=\phi, j=\psi\}}$$

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Viviana Bono, Betti Venneri and Lorenzo Bettini, A typed lambda calculus with intersection types, Theoretical Computer Science: 398:1-3:95–113 (2008)

$$\lambda x^{\phi}.x^{\phi} | \lambda y^{\psi}.y^{\psi}$$

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Andrej Dudenhefner and Jakob Rehof, Intersection type calculi of bounded dimension, POPL 2017: 653–665

$$\vdash_{<>} (\lambda x.x\langle\phi,\psi\rangle)\langle\phi \to \phi, \psi \to \psi\rangle : (\phi \to \phi) \cap (\psi \to \psi)$$

Thank you!