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From Normalization to Typability via Subject Expansion

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λ-Calculus

Definition (λ-Terms)

$$M, N \quad ::= \quad x \mid M\,N \mid \lambda x.M$$

Definition (β-Reduction)

$\rightarrow_\beta$ is the contextual closure of $(\lambda x.M)\,N \rightarrow_\beta M[x := N]$

Definition (Strong Normalization)

M is *strongly normalizing* if each of it's β-reduction paths is finite.

Example

$\lambda y.(\lambda x.y)\,(y\,y)$ is strongly normalizing with the only β-reduction path

$$\lambda y.(\lambda x.y)\,(y\,y) \rightarrow_\beta \lambda y.y$$

Coppo-Dezani Type Assignment System

Definition (Intersection Types)

$$\begin{array}{rcl} A, B & ::= & a \mid \sigma \to A \\ \sigma, \tau & ::= & \{A_1, \ldots, A_n\} \text{ where } n > 0 \end{array}$$

Definition (Type Environments)

$$\Gamma \quad ::= \quad \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$$

Definition (Coppo-Dezani Type Assignment System)

$$\frac{(x : \sigma) \in \Gamma \qquad A \in \sigma}{\Gamma \vdash x : A}\ (\text{Var}) \qquad \frac{\Gamma, x : \sigma \vdash M : A}{\Gamma \vdash \lambda x.M : \sigma \to A}\ (\to\text{I})$$

$$\frac{\Gamma \vdash M : \{A_1, \ldots, A_n\} \to A \qquad \Gamma \vdash N : A_1 \quad \ldots \quad \Gamma \vdash N : A_n}{\Gamma \vdash M\, N : A}\ (\to\text{E})$$

Definition (Typability)

M is *typable* if $\Gamma \vdash M : A$ for some environment Γ and some type A.

Strong Normalization Characterization

Theorem ([Barendregt, Dekkers, and Statman 2013, Theorem 17.2.15 (iii)])

M is strongly normalizing iff M is typable.

Remark ([Barendregt, Dekkers, and Statman 2013, Remark 17.2.16 (ii)])

There are many proofs of Theorem 17.2.15 (iii) in the literature, including

- [Pottinger 1980]
- [Leivant 1986]
- [Van Bakel 1992]
- [Krivine 1990]
- [Ghilezan 1996]
- [Amadio and Curien 1998]

As observed by Venneri (private communication in 1996) all but [Amadio and Curien 1998] contain some bugs.

Strong Normalization Characterization

Theorem

If M is strongly normalizing, then M is typable.

Proof Sketch.

1. Reduce a given M to normal form N
2. Infer typing for N
3. Use (variant of) subject expansion along the reduction path □

Challenges

- Suitable subject expansion statement
- Suitable reduction strategy

First Attempt

Hypothesis

If M is strongly normalizing, $M \rightarrow_\beta N$, and $\Gamma \vdash N : A$, then $\Gamma \vdash M : A$.

Counterexample

Consider $(\lambda x.y)\,(y\,y) \rightarrow_\beta y$

- $\{y : \{a\}\} \vdash y : a$
- $\{y : \{a\}\} \nvdash (\lambda x.y)\,(y\,y) : a$
- $\{y : \{\{a\} \rightarrow a, a\}\} \vdash (\lambda x.y)\,(y\,y) : a$

$\rightsquigarrow$ The type environment may change

Second Attempt

Hypothesis

If M is strongly normalizing, $M \to_\beta N$, and $\Gamma \vdash N : A$, then $\Gamma' \vdash M : A$ for some environment Γ'.

Counterexample

Consider $\lambda y.(\lambda x.y)\,(y\,y) \to_\beta \lambda y.y$

- $\varnothing \vdash \lambda y.y : \{a\} \to a$
- $\Gamma' \nvdash \lambda y.(\lambda x.y)\,(y\,y) : \{a\} \to a$
- $\varnothing \vdash \lambda y.(\lambda x.y)\,(y\,y) : \{\{a\} \to a, a\} \to a$

$\rightsquigarrow$ Not only the environment but the derived type may change

Third Attempt

Proposition

If M is strongly normalizing, $M \to_\beta N$, and N is typable, then M is typable.

Proof.

Easy induction on M. □

Counterargument

Consider the case $M_1\ M_2 \to_\beta M_1\ M_3$ such that

- $\Gamma \vdash M_1\ M_3 : B$
  - $\Gamma \vdash M_1 : \{A\} \to B$
  - $\Gamma \vdash M_3 : A$

⇝ by induction hypothesis $\Gamma' \vdash M_2 : A'$

Unclear how to obtain $\Gamma'' \vdash M_1 : \{A'\} \to B'$?

Fourth Attempt

Proposition

If $M \rightarrow_\beta N$ by contracting the redex $(\lambda x.P)\,Q$, $\Gamma \vdash N : A$, and $\Gamma \vdash Q : B$, then $\Gamma \vdash M : A$.

Counterargument

Consider $\lambda y.(\lambda x.y)\,(y\,y) \rightarrow_\beta \lambda y.y$

- $\Gamma \vdash \lambda y.y : \{a\} \rightarrow a$ for any environment Γ
- $\Gamma \nvdash \lambda y.(\lambda x.y)\,(y\,y) : \{a\} \rightarrow a$ for any environment Γ

$\rightsquigarrow$ above proposition not applicable

Satisfactory Approach

[Amadio and Curien 1998, Lemma 3.5.10]

If $\Gamma \vdash M[x := N] : A$ and $\Gamma \vdash N : B$, then $\Gamma \vdash (\lambda x.M)\, N : A$.

[Amadio and Curien 1998, Theorem 3.5.17]

If M is strongly normalizing, $M \to_\beta N$ by contracting the leftmost redex, and N is typable, then M is typable.

Proof Sketch.

Induction on $(\mathbf{depth}(M), \mathbf{size}(M))$, adjusting derivation above redex. □

Example

Consider $\lambda y.(\lambda x.y)\,(y\,y) \to_\beta \lambda y.y$

- by Lemma 3.5.10 $\{y : \sigma\} \vdash (\lambda x.y)\,(y\,y) : A$
- in proof of Theorem 3.5.17 $\varnothing \vdash \lambda y.(\lambda x.y)\,(y\,y) : \sigma \to A$

Questions

- Is the leftmost reduction strategy necessary?
  - Any perpetual reduction strategy should suffice.
  - Is there strategy-agnostic approach?
- Is strong normalization a *suitable* invariant for subject expansion? Strong normalization immaterial for:
  - Subject expansion at top-level
  - Subject expansion wrt. $\boldsymbol{I}$-reduction
  - Subject expansion wrt. certain cases of $\boldsymbol{K}$-reduction

Definition ($\boldsymbol{I}$-Reduction, $\boldsymbol{K}$-Reduction)

- $\rightarrow_I$ is the contextual closure of $(\lambda x.M)\,N \rightarrow_I M[x := N]$ where $x \in \mathbf{var}(M)$
- $\rightarrow_K$ is the contextual closure of $(\lambda x.M)\,N \rightarrow_K M$ where $x \notin \mathbf{var}(M)$

Subject Expansion wrt. K-reduction

Observation 1 (Top-level K-expansion)

Consider $(\lambda x.M)\,N \to_K M$ such that

- $x \notin \mathrm{var}(M)$
- N is in normal form
- $\Gamma \vdash M : A$

We have $\Gamma' \vdash (\lambda x.M)\,N : A$ for some Γ'.

Remark

In the above observation

- K-expansion into an application
- the derived type A is invariant

Subject Expansion wrt. K-reduction

Observation 2 (Deep K-expansion)

Consider $(\lambda x.M)\, N_1 \to_K (\lambda x.M)\, N_2$ such that

- $N_1 \to_K N_2$
- $x \notin \mathbf{var}(M)$
- N_1 is typable
- $\Gamma \vdash (\lambda x.M)\, N_2 : A$

We have $\Gamma' \vdash (\lambda x.M)\, N_1 : A$ for some Γ'.

Remark

In the above observation

- K-expansion into an application
- the derived type A is invariant

Subject Expansion wrt. K-reduction

Observation 3 (Deep K-expansion)

Consider $x\,M_1 \ldots M_n\,N_1 \to_K x\,M_1 \ldots M_n\,N_2$ such that

- $N_1 \to_K N_2$
- N_1 is typable
- $\Gamma \vdash x\,M_1 \ldots M_n\,N_2 : A$

We have $\Gamma' \vdash x\,M_1 \ldots M_n\,N_1 : A$, adjusting the type of x.

Remark

In the above observation

- K-expansion into an application
- the derived type A is invariant

Subject Expansion wrt. K-reduction

Observation 4 (Deep K-expansion)

Consider $(N_1\ N_2)\ M \to_K N_3\ M$ such that

- $N_1\ N_2 \to_K N_3$ (K-expansion into an application)
- $\Gamma' \vdash N_1\ N_2 : \{A_1, \ldots, A_n\} \to A$ (the derived type is invariant)
- $\Gamma \vdash N_3\ M : A$
  - $\Gamma \vdash N_3 : \{A_1, \ldots, A_n\} \to A$
  - $\Gamma \vdash M : A_1, \ldots, \Gamma \vdash M : A_n$

We have $\Gamma'' \vdash (N_1\ N_2)\ M : A$, combining Γ and Γ' into Γ''.

Remark

In the above observation

- K-expansion into an application
- the derived type A is invariant

Subject Expansion wrt. K-reduction

Observation 5 (K-expansion into an abstraction)

Consider $\lambda x.M \to_K \lambda x.N$ such that

- $M \to_K N$
- $\Gamma \vdash N : A$
- $\Gamma' \vdash M : B$

We have $\Gamma'' \vdash \lambda x.M : \sigma \to B$ for suitable σ.

Remark

In the above observation

- K-expansion into an abstraction
- the derived type may change

Combining Observations

Definition (K'-Reduction)

1. If N is in β-normal form and $x \notin \mathbf{var}(M)$, then $(\lambda x.M)\, N \to_{K'} M$.
2. If $N_1 \to_{K'} N_2$, then $x\, M_1 \ldots M_n\, N_1 \to_{K'} x\, M_1 \ldots M_n\, N_2$ where $n \geq 0$.
3. If $N_1 \to_{K'} N_2$ and $x \notin \mathbf{var}(M)$, then $(\lambda x.M)\, N_1 \to_{K'} (\lambda x.M)\, N_2$.
4. If $M_1\, M_2 \to_{K'} M_3$, then $M_1\, M_2\, N \to_{K'} M_3\, N$.
5. If $M \to_{K'} N$, then $\lambda x.M \to_{K'} \lambda x.N$.

Lemma (Subject Expansion wrt. $\to_{K'}$)

If $M \to_{K'} N$ and $\Gamma \vdash N : A$, then

- *if M is an application, then $\Gamma' \vdash M : A$ for some environment Γ'*
- *if M is an abstraction, then M is typable*

Proof.

Induction on the definition of $\to_{K'}$, using Observations (1) – (5). □

- Mechanized using the Coq proof assistant.

$\boldsymbol{IK'}$-Reduction Properties

Definition ($\boldsymbol{IK'}$-Reduction)

$\rightarrow_{\boldsymbol{IK'}}$ denotes the union of $\rightarrow_{\boldsymbol{I}}$ and $\rightarrow_{\boldsymbol{K'}}$.

Lemma (Dudenhefner and Pautasso 2024, Lemma 5)

If $\boldsymbol{M} \rightarrow_{\beta} \boldsymbol{N}$, then there exists $\boldsymbol{N'}$ such that $\boldsymbol{M} \rightarrow_{\boldsymbol{IK'}} \boldsymbol{N'}$.

Corollary

If $\boldsymbol{M}$ is strongly normalizing, then there exists $\boldsymbol{N}$ in β-normal form such that $\boldsymbol{M} \rightarrow^{*}_{\boldsymbol{IK'}} \boldsymbol{N}$.

Corollary

If $\boldsymbol{M}$ is strongly normalizing, then $\boldsymbol{M}$ is typable.

Further Implications

Theorem

If $\boldsymbol{M} \rightarrow_{\boldsymbol{IK'}} \boldsymbol{N}$ and $\boldsymbol{N}$ is strongly normalizing, then $\boldsymbol{M}$ is strongly normalizing.

Perpetual Reduction Strategies

Definition (F_∞ [Barendregt 1985, Definition 13.4.1])

$$F_\infty(M) = \begin{cases} M \text{ if } M \text{ is in normal form} \\ \text{If } M = C[(\lambda x.P)\,Q] \text{ and } (\lambda x.P)\,Q \text{ is the leftmost redex, then} \\ \begin{cases} C[P[x := Q]] \text{ if } x \in \mathbf{var}(P) \\ C[P] \text{ if } x \notin \mathbf{var}(P) \text{ and } Q \text{ is in normal form} \\ C[(\lambda x.P)\,F_\infty(Q)] \text{ otherwise} \end{cases} \end{cases}$$

Lemma

$M \to^*_{IK'} F_\infty(M)$

Corollary (F_∞ is perpetual [Barendregt 1985, Theorem 13.4.6])

If $F_\infty(M)$ is strongly normalizing, then M is strongly normalizing.

Syntactic Proofs of Strong Normalization

Definition (γ-Reduction [Kfoury and Wells 1995, Definition 3.1])

$\to_\gamma$ is the contextual closure of $\big(\lambda x.(\lambda y.M)\big)\, N \to_\gamma \lambda y.(\lambda x.M)\, N$

Lemma (Subject Expansion wrt. $\to_\gamma$)

If $M \to_\gamma N$ and $\Gamma \vdash N : A$, then $\Gamma \vdash M : A$.

Corollary ([Kfoury and Wells 1995, Theorem 3.11])

If $M \to^*_{I\gamma} P \to^*_K N$ such that N is in β-normal form, then M is strongly normalizing.

Type Inference in Quantitative Type Systems

Corollary ([Dudenhefner and Pautasso 2024, Theorem 32])

If M is typable in the simply typed λ-calculus, then M is typable in the uniform non-idempotent type system.

Proof.

1. Reduce M to a normal form N using $IK'\gamma$-reduction.
2. Infer a uniform intersection type for N.
3. Construct a uniform intersection type for M via subject expansion. □

Remark

Subject expansion properties for $IK'(\gamma)$-reduction hold in various (non-idempotent) intersection type systems.

Closing Remarks

- Historically, typability proofs for strongly normalizing terms are inaccurate.
- $\boldsymbol{IK'}$-reduction is a (sufficiently) large fragment of β-reduction.
- $\boldsymbol{IK'}(\gamma)$-reduction enjoys subject expansion in intersection type systems.
  - No need for leftmost reduction strategies.
  - Strong normalization is not an explicit requirement.
- $\rightsquigarrow$ $\boldsymbol{IK'}(\gamma)$-reduction is suitable to transport typability between type systems.
- $\rightsquigarrow$ Any $\boldsymbol{IK'}(\gamma)$-reduction strategy is perpetual.
  - Weak $\boldsymbol{IK'}(\gamma)$-normalization implies strong β-normalization.
- $\boldsymbol{IK'}(\gamma)$-reduction discovered by interaction with a proof assistant.

Lemma (Subject Expansion wrt. $\rightarrow_{K'}$)

If $M \rightarrow_{K'} N$ and $\Gamma \vdash N : A$, then

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- Is $\boldsymbol{IK'}(\gamma)$-reduction (in some sense) maximal?

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Thank You!