## Separating Terms through Multi Types

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When studying program equivalence, one searches for any small hint that programs are not equivalent:

 $\lambda x.x + x$  and  $\lambda x.x * x$  can be distinguished by the input 1 (but not by the input 2)

 $\lambda x.x + x$  and  $\lambda x.$ "Hello World" can be distinguished by types:  $\lambda x.x + x:$  int  $\rightarrow$  int and  $\lambda x.$ "Hello World" :  $\alpha \rightarrow$  string

## Separating Terms

Let t and u two  $\lambda$ -terms.

- Does there exist a context C such that C(t) does not terminate and C(u) terminates?
- Does there exist a typing judgment (Γ, L) such that Γ ⊢ t:L but Γ ∀ u:L?

This talk is about separating with Intersection Types!

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This talk is about separating with Intersection Types!

Separating Terms with Contexts: let's reduce term to their head normal form and find an appropriate context.

 $\blacktriangleright$  I and  $\Omega$  are separated by the empty context  $\langle\cdot\rangle$ 

►  $x \text{ II and } x \Omega \text{ I}$  are separated by  $(\lambda x. \langle \cdot \rangle) \pi_1$  where  $\pi_1 = \lambda x. \lambda y. x$ 

▶  $y I (x \Omega I)$  and y I (x I I) are separated by  $(\lambda y.(\lambda x.\langle \cdot \rangle)\pi_1)\pi_2$ where  $\pi_2 = \lambda x.\lambda y.y$ 

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Separating Terms with Types: easier than context Böhm out to select subterms!

 $x I (x \Omega I) \text{ and } x I (x I I)$  $x: [[A] \multimap [B] \multimap B, [A] \multimap [B] \multimap A] \vdash : B$  $\Gamma \vdash x I (x I I): B \text{ but } \Gamma \nvDash x I (x \Omega I): B$ 

Separating Terms with Types: easier than context Böhm out to select subterms!

## This Talk

For Weak Head Call-by-Name:

- A coinductive flavor (normal form bisimulations): no approximants
- Derivation transfer requires non idempotent intersection types

Weak Head Multi Types

LINEAR TYPES  $L, L' ::= \star_k | M \multimap L \quad k \in \mathbb{N}$ MULTI TYPES  $M, N ::= [L_1, \dots, L_n] \quad n \ge 0$  $\frac{1}{x:[L] \vdash x:L} \text{ ax } \frac{\emptyset \vdash \lambda x.t:\star_k}{\emptyset \vdash \lambda x.t:\star_k} \lambda_k \quad \frac{\Gamma, x:M \vdash t:L}{\Gamma \vdash \lambda x.t:M \multimap L} \lambda$   $\frac{\Gamma \vdash t:[L_i]_{i \in I} \multimap L' \quad (\Gamma_i \vdash u:L_i)_{i \in I} \quad I \text{ finite}}{\Gamma \uplus \Delta \vdash tu:L'} @$ 

## Type Equivalence

**Type Equivalence:**  $t \simeq_{type} u$  if  $\forall (\Gamma, L) \ \Gamma \vdash t : L \iff \Gamma \vdash u : L$ 

 $\beta$ -equivalence is included in  $\simeq_{type}$ :

by Subject Reduction and Expansion

 $\eta$ -equivalence is not included in  $\simeq_{type}$ :

• 
$$x: [\star_0] \vdash x: \star_0$$
 but  $x: [\star_0] \not\vdash \lambda y. x y: \star_0$ 

• 
$$\emptyset \vdash \lambda y.x y: \star_0$$
 but  $\emptyset \not\vdash x: \star_0$ 

## Normal form bisimilarity

The syntactical characterization is phrased in a coinductive way, using Sangiorgi's normal form bisimilarity. It coincides with the inequational theory induced by Lévy-Longo trees.

### Definition (Weak head normal form similarity)

A relation  $\mathcal{R}$  is a **weak head normal (whnf) form simulation** if whenever  $t \mathcal{R} u$  holds, we have that either:

(
$$\perp$$
)  $t \not \Downarrow_{wh}$ , or,  
(abs)  $t \not \Downarrow_{wh} \lambda x.t'$  and  $u \not \Downarrow_{wh} \lambda x.u'$  with  $t' \mathcal{R} u'$ , or,  
(app)  $t \not \Downarrow_{wh} y t_1 \cdots t_k$  and  
 $u \not \Downarrow_{wh} y u_1 \cdots u_k$  with  $(t_i \mathcal{R} u_i)_{i \le k}$ .

Whnf similarity, noted  $\precsim_{\rm nf}$ , is the largest whnf simulation.

## Derivation Transfer

From bisimilar terms to type equivalent types

#### 

#### The proof goes by induction on the size of derivations.

Assume  $t \preceq_{\text{nf}} u$  and  $\pi \triangleright \Gamma \vdash t : L$ .

By Quantitative Subject Reduction,  $\pi' \triangleright \Gamma \vdash n_t : L \text{ s.t. } |\pi'| \le |\pi|$ . By case analysis on the shape of  $n_t$ :

#### **Application case:**

$$\pi': \frac{\Gamma \vdash xt_1 \cdots t_{k-1} : [L_i]_{i \in I} \multimap L \quad (\Delta_i \vdash t_k : L_i)_{i \in I}}{\Gamma \uplus (\uplus_{i \in I} \Delta_i) \vdash n_t = xt_1 \cdots t_k : L} @$$
  
$$\rightsquigarrow \quad \sigma \triangleright \Gamma \vdash n_u : L \quad \text{as } xt_1 \cdots t_{k-1} \precsim_{nf} xu_1 \cdots u_{k-1} \text{ and } t_k \precsim_{nf} u_k.$$
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## Type Equivalence & Compositionality

#### Type equivalence is **compositional**.

- ▶  $t \preceq_{type} u$  and  $s \preceq_{type} r$  implies  $ts \preceq_{type} ur$
- $t \preceq_{\text{type}} u$  implies  $\lambda x.t \preceq_{\text{type}} \lambda x.u$

Proposition (Soundness wrto Contextual Equivalence) If  $t \preceq_{type} u$  then  $t \leq_{C}^{wh} u$ 

Side note:  $t \leq_{C}^{wh} u$  does not imply  $t \gtrsim_{type} u$  (problems with  $\eta...$ )

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## Type-Böhm Out

Separating un-bisimilar terms

## NF Bisimilarity \_\_\_\_\_ Type-Böhm Out \_\_\_\_\_ Type Equivalence

**Coinduction principle:**  $\leq_{type}$  is a whnf simulation  $\Rightarrow \leq_{type} \subseteq \leq_{nf}$ 

We show that  $\preceq_{\text{type}}$  is a whnf simulation by contrapositive: Let *t* and *u* such that  $t \downarrow_{wh} n_t$  and  $u \downarrow_{wh} n_u$  with **different** normal forms

→ Build a Separating Type

### Separating Terms through Multi Types Some examples

1. Separating any abstraction  $\lambda x.t$  and any applied of  $x t_1 \cdots t_k$ :

- 2. Separating a variable x and an applied nf xt:
- 3. Separating different abstractions:

,

4. Separating applied nf where arguments differ, say xt and xu:

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$$\begin{array}{c|c} \hline \Gamma \vdash x : \star_0 \\ \hline \Gamma \vdash x : \star_0 \end{array} & \begin{array}{c} \vdots \\ \hline \Gamma \vdash x t : \star_0 \\ \hline \end{array} \\ \hline \\ \Gamma = x : [\star_0] \end{array} & \begin{array}{c} \Gamma \supseteq x : [M \multimap \cdots, \ldots] \end{array}$$

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Suppose we have  $\Gamma, x: M \vdash t: L$  but  $\Gamma, x: M \nvDash u: L$ 

$$\begin{array}{c|c} \hline{\Gamma, x: M \vdash t: L} \\ \hline{\Gamma \vdash \lambda x. t: M \multimap L} \end{array} \lambda \\ \hline \text{If} \quad \overline{\Gamma \vdash \lambda x. u: M \multimap L} \\ \hline \text{Then it must start with:} \\ \hline \hline{\Gamma \vdash \lambda x. u: M \multimap L} \end{array} \lambda$$

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- 4. Separating applied nf where arguments differ, say xt and xu:

Suppose we have  $\Gamma \vdash t: L$  but  $\Gamma \not\vdash u: L$  $\frac{\Delta \vdash x: [L] \multimap \star_k \quad \Gamma \vdash t: L}{\Gamma \uplus \Delta \vdash x \, t: \star_i} \qquad \begin{vmatrix} \mathbf{I} \mathbf{f} & \frac{\vdots}{\Gamma \vdash x \, u: \star_i} \\ \mathbf{I} \mathbf{f} & \frac{\Box \vdash x: \star_i}{\Gamma \vdash x \, u: \star_i} \\ \frac{\Delta' \vdash x: [L_i]_i \multimap \star_k \quad (\Gamma_i \vdash u: L_i)_i}{\Gamma \uplus \Delta' \vdash x \, u: \star_i} \\ \end{bmatrix}$ If k is well chosen, then  $\Delta' = x: [[L] \multimap \star_k]$ 

## Factorizing the need of countable atoms

# Compositionality of $\not \gtrsim_{type}$ is the only part that requires many distinguishable ground types!

### Intuitively: $t \not\gtrsim_{type} u$ or $s \not\gtrsim_{type} r$ implies $ts \not\gtrsim_{type} ur$

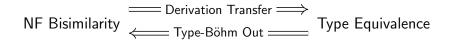
- ► Left difference:  $x t_1 \cdots t_k \gtrsim_{type} y u_1 \cdots u_{k'}$  implies  $x t_1 \cdots t_k s \gtrsim_{type} y u_1 \cdots u_{k'} r$ .
- ► Right difference:  $x t_1 \cdots t_k \gtrsim_{type} y u_1 \cdots u_{k'}$  and  $s \gtrsim_{type} r$ implies  $x t_1 \cdots t_k s \gtrsim_{type} y u_1 \cdots u_k r$ .

## Factorizing the need of countable atoms

Intuitively:  $t \not\preceq_{type} u$  or  $s \not\preceq_{type} r$  implies  $ts \not\preceq_{type} ur$ 

- ► Left difference:  $x t_1 \cdots t_k \not \subset_{\text{type}} y u_1 \cdots u_{k'}$  implies  $x t_1 \cdots t_k s \not \subset_{\text{type}} y u_1 \cdots u_{k'} r$ .
- ► Right difference:  $x t_1 \cdots t_k \preccurlyeq_{type} y u_1 \cdots u_{k'}$  and  $s \preccurlyeq_{type} r$  implies  $x t_1 \cdots t_k s \preccurlyeq_{type} y u_1 \cdots u_k r$ .

## Conclusion

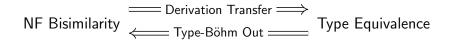


#### **Perspectives:**

- Towards Call-by-Value equivalences, where NF bisimilarities are more involved
- Can we adapt type equivalence to reach full abstraction?

Can Type-Böhm out help to simplify (Context-)Böhm out ? Thank you!

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### Thank you!