Intersection Types Meet Session Types

Reconciling Two Different Views on Computational Resources

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UNIFYING CORRECTNESS FOR COMMUNICATING SOFTWARE

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This Talk

A brief overview of formal connections between intersection types with session types for concurrency (FSCD'21, TYPES'21, APLAS'23, MFPS'24).

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- Non-idempotent intersection types and session types with Curry-Howard foundations (aka 'propositions-as-sessions')
- Two different approaches on resource control:
  - In λ: Resources are terms, whose types can ensure quantitative properties
  - In π: Resources are channels, whose types enforce linear usage for correctness
- Key unifying aspects: non-determinism, confluence / commitment, failures

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- Relative expressiveness: Typed λ-calculi translated into session-typed π-calculi

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- Key unifying aspects: non-determinism, confluence / commitment, failures
- Relative expressiveness: Typed λ-calculi translated into session-typed π-calculi
- A concurrent interpretation of intersection types, with strong correctness properties: static (type preservation) and dynamic (operational correspondence)

Our Translation, In a Nutshell (FSCD'21 / LMCS'23)

Source: $\lambda^{\lightning}_{\oplus}$, a resource λ-calculus with non-determinism and failure

- Applications $M\ B$, where B is a bag of terms, to be non-determically fetched
- Failure occurs when a bag's size doesn't match the variable occurrences
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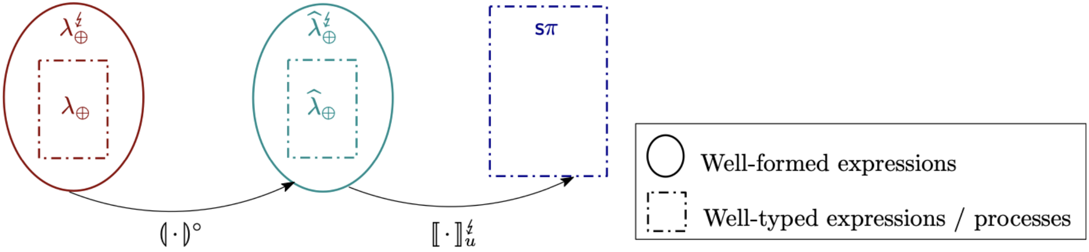
Target: sπ, a session-typed π-calculus with non-deterministically available behaviors

- **Session types**: The protocols that names of a process must respect
- Under 'propositions-as-sessions', session type = linear logic proposition
- Non-deterministic processes, whose protocols may proceed as intended or fail
- Well-typed processes respect their types, with confluence and deadlock-freedom

Our Translation, In a Nutshell (FSCD'21 / LMCS'23)

Structly speaking, two translations:

1. We first translate $\lambda^{\lightning}_{\oplus}$ into a resource λ-calculus with sharing constructs
   A direct translation, useful to make different variable occurrences explicit
2. The translation into processes leverages sharing to enforce linearity; failure for terms accounted as non-available protocols



A resource λ-calculus with failure ($\lambda^{\lightning}_{\oplus}$)

Syntax:

$$
\begin{array}{llcl}
\text{(Terms)} & M, N, L & ::= & x \mid \lambda x.M \mid (M\ B) \mid M\langle\!\langle B/x \rangle\!\rangle \mid \mathtt{fail}^{\widetilde{x}} \\
\text{(Bags)} & A, B & ::= & 1 \mid \wr\, M \,\int \mid A \cdot B \\
\text{(Expressions)} & \mathbb{M}, \mathbb{N}, \mathbb{L} & ::= & M \mid \mathbb{M} + \mathbb{N}
\end{array}
$$

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\end{array}
$$

Reduction (Excerpt):

$$
[\mathtt{R : Beta}]\ \frac{}{(\lambda x.M)B \longrightarrow M\ \langle\!\langle B/x \rangle\!\rangle}
$$

$$
[\mathtt{R : Fetch}]\ \frac{\mathsf{head}(M) = x \qquad B = \wr N_1, \ldots, N_k \wr\ ,\ k \geq 1 \qquad \#(x, M) = k}{M\ \langle\!\langle B/x \rangle\!\rangle \longrightarrow M\{\!| N_1/x |\!\}\langle\!\langle (B \setminus\!\setminus N_1)/x \rangle\!\rangle + \cdots + M\{\!| N_k/x |\!\}\langle\!\langle (B \setminus\!\setminus N_k)/x \rangle\!\rangle}
$$

$$
[\mathtt{R : Fail}]\ \frac{\#(x, M) \neq \mathsf{size}(B) \qquad \widetilde{y} = (\mathsf{mfv}(M) \setminus x) \uplus \mathsf{mfv}(B)}{M\ \langle\!\langle B/x \rangle\!\rangle \longrightarrow \sum_{\mathsf{PER}(B)} \mathtt{fail}^{\widetilde{y}}}
$$

Intersection Types

We define strict and multiset types by the grammar:

$$\begin{array}{lrcl} \text{(Strict)} & \sigma, \tau, \delta & ::= & \mathbf{unit} \mid \pi \to \sigma \\ \text{(Multiset)} & \pi & ::= & \sigma^k \mid \omega \end{array}$$

where σ^k stands for the intersection $\sigma \wedge \cdots \wedge \sigma$ (k times, for some $k > 0$)

Given **type contexts** $\Gamma, \Delta, \ldots$ (sets of type assignments $x : \pi$), **type judgements** are of the form

$$\Gamma \vdash \mathbb{M} : \sigma$$

Well-Typed $\neq$ Well-Formed

- **Well-typed** terms use resources properly

$$[\mathtt{T} : \mathtt{app}] \; \frac{\Gamma \vdash M : \pi \to \tau \qquad \Delta \vdash B : \pi}{\Gamma \wedge \Delta \vdash M\ B : \tau} \qquad [\mathtt{T} : \mathtt{ex\text{-}sub}] \; \frac{\Gamma, x : \sigma^k \vdash M : \tau \qquad \Delta \vdash B : \sigma^k}{\Gamma \wedge \Delta \vdash M \langle\!\langle B/x \rangle\!\rangle : \tau}$$

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- **Well-formed** terms allow resource mismatches, which lead to failure

$$[\mathsf{F:ex\text{-}sub}]\ \frac{\Gamma, x : \sigma^k \models M : \tau \qquad \Delta \models B : \sigma^j \quad k, j \geq 0}{\Gamma \wedge \Delta \models M\langle\!\langle B/x \rangle\!\rangle : \tau}$$

$$[\mathsf{F:app}]\ \frac{\Gamma \models M : \sigma^j \to \tau \qquad \Delta \models B : \sigma^k \qquad k, j \geq 0}{\Gamma \wedge \Delta \models M\ B : \tau}$$

Well-typed processes are also well-formed (but not the other way around)

Sharing is Caring

- In the sharing calculus, denoted $\widehat{\lambda}^{\lightning}_{\oplus}$, a variable can only appear once in a term
- Multiple occurrences of the same variable are “atomized”
- Sharing of variables $\widetilde{x}$ occurring in M using x:

$$M[\widetilde{x} \leftarrow x]$$

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$$M[\widetilde{x} \leftarrow x]$$

- Examples:

$$\lambda x.x_1[x_1 \leftarrow x] \qquad \lambda x.x_1 \wr x_2 \int [x_1, x_2 \leftarrow x]$$

- The shared variables $\widetilde{x}$ can be empty:
  $M[\leftarrow x]$ says that x does not share any variables in M.

A π-calculus with non-deterministic sessions (sπ)

Fragment from Caires and Pérez (ESOP'21):

$$
\begin{array}{rlll}
P, Q ::= & \overline{x}[y].(P \mid Q) & \mid & x(y).P \\
\mid & x.\overline{\texttt{some}}; P & \mid & x.\texttt{some}_{w_1,\ldots,w_n}; P \\
\mid & x.\overline{\texttt{none}} & & \\
\mid & \overline{x}[\,] & \mid & x(\,).P \\
\mid & (\boldsymbol{\nu} x)(P \mid Q) & & \\
\mid & P \oplus Q & \mid & P \mid Q \\
\mid & 0 & \mid & [x \leftrightarrow y]
\end{array}
$$

Confluent Non-determinism and Failures in sπ

$$y_1(z).y_2(w).0$$

Confluent Non-determinism and Failures in sπ

$$x.\mathtt{some}_{(y_1, y_2)}; y_1(z).y_2(w).0$$

Confluent Non-determinism and Failures in sπ

$$R = \; (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; y_1(z).y_2(w).0 \mid (x.\overline{\mathtt{some}}; P \oplus x.\overline{\mathtt{none}}))$$

Confluent Non-determinism and Failures in sπ

$$
\begin{aligned}
R = \;& (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; y_1(z).y_2(w).0 \mid (x.\overline{\mathtt{some}}; P \oplus x.\overline{\mathtt{none}})) \\
\equiv \;& (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; y_1(z).y_2(w).0 \mid x.\overline{\mathtt{some}}; P) \\
& \qquad\qquad\qquad\oplus \\
& (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; y_1(z).y_2(w).0 \mid x.\overline{\mathtt{none}})
\end{aligned}
$$

Confluent Non-determinism and Failures in sπ

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\equiv {} & (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; y_1(z).y_2(w).0 \mid x.\overline{\mathtt{some}}; P) \\
& \qquad\qquad\qquad\qquad \oplus \\
& (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; y_1(z).y_2(w).0 \mid x.\overline{\mathtt{none}})
\end{aligned}
$$

Letting $Q = y_1(z).y_2(w).0$, we have:

$$
\begin{tikzcd}
 & (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; Q \mid x.\overline{\mathtt{some}}; P) \oplus (y_1.\overline{\mathtt{none}} \mid y_2.\overline{\mathtt{none}}) \arrow[dr] & \\
R = (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; Q \mid (x.\overline{\mathtt{some}}; P \oplus x.\overline{\mathtt{none}})) \arrow[ur] \arrow[dr] & & (\nu x)(Q \mid P) \oplus (y_1.\overline{\mathtt{none}} \mid y_2.\overline{\mathtt{none}}) \\
 & (\nu x)(Q \mid P) \oplus (\nu x)(x.\mathtt{some}_{(y_1,y_2)}; Q \mid x.\overline{\mathtt{none}}) \arrow[ur] &
\end{tikzcd}
$$

Session Types for sπ

Linear logic propositions as session types (cf. Caires & Pfenning, Wadler):

$$
\begin{array}{rll}
A, B ::= & \bot & \text{(closed session)} \\
 & \mid 1 & \text{(closed session)} \\
 & \mid A \otimes B & \text{(output } A\text{, continue as } B\text{)} \\
 & \mid A \parr B & \text{(input } A\text{, continue as } B\text{)} \\
 & \mid \& A & \text{(may produce } A\text{)} \\
 & \mid \oplus A & \text{(may consume } A\text{)}
\end{array}
$$

A duality relation on types/propositions connects complementary behaviors

The Translation $[\![\cdot]\!]_u : \widehat{\lambda}^{\lightning}_{\oplus} \to \mathsf{s}\pi$

$$
\begin{aligned}
[\![x]\!]_u &= x.\overline{\mathtt{some}}; [x \leftrightarrow u] \\
[\![\lambda x.M[\tilde{x} \leftarrow x]]\!]_u &= u.\overline{\mathtt{some}}; u(x).[\![M[\tilde{x} \leftarrow x]]\!]_u \\
[\![M(\wr N_1 \int \cdot \wr N_2 \int)]\!]_u &= (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{fv}(B)}; \\
&\qquad\qquad \overline{v}[x].([v \leftrightarrow u] \mid [\![(\wr N_1 \int \cdot \wr N_2 \int)]\!]_x)) \\
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&\quad (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{fv}(B)}; \\
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[\![M[\tilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle]\!]_u &= \bigoplus_{B_i \in \mathsf{PER}(B)} (\nu x)([\![M[\tilde{x} \leftarrow x]]\!]_u \mid [\![B_i]\!]_x)
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$$\llbracket x \rrbracket_u = x.\overline{\mathtt{some}}; [x \leftrightarrow u]$$

$$\llbracket \lambda x.M[\widetilde{x} \leftarrow x] \rrbracket_u = u.\overline{\mathtt{some}}; u(x).\llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u$$

$$\begin{aligned}\llbracket M(\wr N_1 \int \cdot \wr N_2 \int) \rrbracket_u = {} & (\nu v)(\llbracket M \rrbracket_v \mid v.\mathtt{some}_{u,\mathsf{fv}(B)}; \\ & \qquad \overline{v}[x].([v \leftrightarrow u] \mid \llbracket (\wr N_1 \int \cdot \wr N_2 \int) \rrbracket_x)) \\ & \oplus \\ & (\nu v)(\llbracket M \rrbracket_v \mid v.\mathtt{some}_{u,\mathsf{fv}(B)}; \\ & \qquad \overline{v}[x].([v \leftrightarrow u] \mid \llbracket (\wr N_2 \int \cdot \wr N_1 \int) \rrbracket_x))\end{aligned}$$

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\mid x.\overline{\mathtt{some}}; x.\mathtt{some}_{u,(\mathsf{fv}(M)\setminus x_1,\cdots,x_n)}; x(x_1). \\
x.\overline{\mathtt{some}}.\overline{x}[y_2].(y_2.\mathtt{some}_{\emptyset}; y_2(\,); 0 \\
\mid x.\overline{\mathtt{some}}; x.\mathtt{some}_{u,(\mathsf{fv}(M)\setminus x_2)}; x(x_2) \\
.x.\overline{\mathtt{some}}; \overline{x}[y_3].(y_3.\mathtt{some}_{u,\mathsf{fv}(M)}; y_3(\,); [\![M]\!]_u \mid x.\overline{\mathtt{none}})\;)) \\
[\![M[\leftarrow x]]\!]_u = x.\overline{\mathtt{some}}.\overline{x}[y_i].(y_i.\mathtt{some}_{u,\mathsf{fv}(M)}; y_i(\,); [\![M]\!]_u \mid x.\overline{\mathtt{none}})
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$$

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$$
\begin{aligned}
[\![\wr M \int \cdot B]\!]_x = {} & x.\mathtt{some}_{\mathsf{fv}(\wr M \int \cdot B)}; x(y_i).x.\mathtt{some}_{y_i,\mathsf{fv}(\wr M \int \cdot B)}; x.\overline{\mathtt{some}}; \overline{x}[x_i] \\
& .(x_i.\mathtt{some}_{\mathsf{fv}(M)}; [\![M]\!]_{x_i} \mid [\![B]\!]_x \mid y_i.\overline{\mathtt{none}}) \\
[\![1]\!]_x = {} & x.\mathtt{some}_{\emptyset}; x(y_n).(y_n.\overline{\mathtt{some}}; \overline{y_n}[\,] \mid x.\mathtt{some}_{\emptyset}; x.\overline{\mathtt{none}}) \\
[\![\mathbb{M} + \mathbb{N}]\!]_u = {} & [\![\mathbb{M}]\!]_u \oplus [\![\mathbb{N}]\!]_u \\
[\![\mathtt{fail}^{x_1,x_2}]\!]_u = {} & u.\overline{\mathtt{none}} \mid x_1.\overline{\mathtt{none}} \mid x_2.\overline{\mathtt{none}}
\end{aligned}
$$

The Translation at Work

$$
\begin{tikzcd}
(\lambda a.a) \wr M \int \arrow[d] \\
a \langle\!\langle {}^{\wr M \int}/a \rangle\!\rangle \arrow[d] \\
M \langle\!\langle {}^{1}/a \rangle\!\rangle \arrow[d, "\leq"] \\
M
\end{tikzcd}
$$

The Translation at Work

$$
\begin{tikzcd}
(\lambda a. a_1[a_1 \leftarrow a]) \lbag M \rbag \arrow[d] \\
a_1[a_1 \leftarrow a]\langle\!\langle {}^{\lbag M \rbag}/_{a} \rangle\!\rangle \arrow[d] \\
M[\leftarrow a]\langle\!\langle {}^{1}/_{a} \rangle\!\rangle \arrow[d, "\leq"] \\
M
\end{tikzcd}
$$

The Translation at Work

$$
\begin{tikzcd}
(\lambda a.a_1[a_1 \leftarrow a]) \wr M \int \arrow[r, "{\llbracket \cdot \rrbracket_u}"] \arrow[d] & \begin{array}{l}(\nu v)(v.\overline{\mathtt{some}}; v(x).\llbracket a_1[a_1 \leftarrow a] \rrbracket_v \\ \qquad \mid v.\mathtt{some}_{u,b}; \overline{v}[x].([v \leftrightarrow u] \mid \llbracket \wr M \int \rrbracket_x)\end{array} \\
a_1[a_1 \leftarrow a]\langle\!\langle {}^{\wr M \int}/a \rangle\!\rangle \arrow[d] & \\
M[\leftarrow a]\langle\!\langle {}^{1}/a \rangle\!\rangle \arrow[d, "\leq"] & \\
M &
\end{tikzcd}
$$

The Translation at Work

$$
\begin{tikzcd}[column sep=large]
(\lambda a.a_1[a_1 \leftarrow a]) \wr M \int \arrow[r, "\llbracket \cdot \rrbracket_u"] \arrow[d] & \begin{array}{l}(\nu v)(v.\overline{\texttt{some}}; v(x).\llbracket a_1[a_1 \leftarrow a] \rrbracket_v \\ \quad \mid v.\texttt{some}_{u,b}; \overline{v}[x].([v \leftrightarrow u] \mid \llbracket \wr M \int \rrbracket_x)\end{array} \arrow[d] \\
a_1[a_1 \leftarrow a]\langle\langle {}^{\wr M \int}/a \rangle\rangle \arrow[r, "\llbracket \cdot \rrbracket_u"] \arrow[d] & (\nu a)(\llbracket a_1[a_1 \leftarrow a] \rrbracket_u \mid \llbracket \wr M \int \rrbracket_a) \\
M[\leftarrow a]\langle\langle {}^{1}/a \rangle\rangle \arrow[d, "\leq"] & \\
M &
\end{tikzcd}
$$

The Translation at Work

$$
\begin{tikzcd}
(\lambda a.a_1[a_1 \leftarrow a]) \wr M \wr \arrow[r, "{[\![\cdot]\!]_u}"] \arrow[d] & \begin{array}{l}(\nu v)(v.\overline{\mathtt{some}}; v(x).[\![a_1[a_1 \leftarrow a]]\!]_v \\ \quad \mid v.\mathtt{some}_{u,b}; \overline{v}[x].([v \leftrightarrow u] \mid [\![\wr M \wr]\!]_x)\end{array} \arrow[d] \\
a_1[a_1 \leftarrow a]\langle\!\langle \wr M \wr / a \rangle\!\rangle \arrow[r, "{[\![\cdot]\!]_u}"] \arrow[d] & (\nu a)([\![a_1[a_1 \leftarrow a]]\!]_u \mid [\![\wr M \wr]\!]_a) \arrow[dd] \\
M[\leftarrow a]\langle\!\langle 1/a \rangle\!\rangle \arrow[d, "\leq"] & \\
M \arrow[r, "{[\![\cdot]\!]_u}"] & [\![M]\!]_u
\end{tikzcd}
$$

The Translation at Work: Failure

$$
\begin{array}{c}
(\lambda a.a)\mathtt{1} \\
\big\downarrow \\
a\langle\!\langle \mathtt{1}/a \rangle\!\rangle \\
\big\downarrow \\
\mathtt{fail}^{\emptyset}
\end{array}
$$

The Translation at Work: Failure

$$(\lambda a.a_1[a_1 \leftarrow a])\mathtt{1}$$

$$\downarrow$$

$$a_1[a_1 \leftarrow a]\langle\langle \mathtt{1}/a \rangle\rangle$$

$$\downarrow$$

$$\mathtt{fail}^{\emptyset}$$

The Translation at Work: Failure

$$
\begin{tikzcd}
(\lambda a. a_1[a_1 \leftarrow a])1 \arrow[r, "\llbracket \cdot \rrbracket_u"] \arrow[d] & (\nu v)(v.\overline{\texttt{some}}; v(x).\llbracket a_1[a_1 \leftarrow a] \rrbracket_v \mid v.\texttt{some}_{u,b}; \overline{v}[x].([v \leftrightarrow u] \mid \llbracket 1 \rrbracket_x)) \\
a_1[a_1 \leftarrow a]\langle\!\langle 1/a \rangle\!\rangle \arrow[d] & \\
\texttt{fail}^{\emptyset} &
\end{tikzcd}
$$

The Translation at Work: Failure

$$
\begin{tikzcd}
(\lambda a.a_1[a_1 \leftarrow a])1 \arrow[r, "\llbracket \cdot \rrbracket_u"] \arrow[d] & (\nu v)(v.\overline{\mathtt{some}}; v(x).\llbracket a_1[a_1 \leftarrow a] \rrbracket_v \mid v.\mathtt{some}_{u,b}; \overline{v}[x].([v \leftrightarrow u] \mid \llbracket 1 \rrbracket_x)) \arrow[d] \\
a_1[a_1 \leftarrow a]\langle\!\langle 1/a \rangle\!\rangle \arrow[r, "\llbracket \cdot \rrbracket_u"] \arrow[d] & (\nu a)(\llbracket a_1[a_1 \leftarrow a] \rrbracket_u \mid \llbracket 1 \rrbracket_a) \\
\mathtt{fail}^{\emptyset} &
\end{tikzcd}
$$

The Translation at Work: Failure

$$
\begin{tikzcd}
(\lambda a.a_1[a_1 \leftarrow a])1 \arrow[r, "{[\![\cdot]\!]_u}"] \arrow[d] & (\nu v)(v.\overline{\mathtt{some}}; v(x).[\![a_1[a_1 \leftarrow a]]\!]_v \mid v.\mathtt{some}_{u,b}; \overline{v}[x].([v \leftrightarrow u] \mid [\![1]\!]_x)) \arrow[d] \\
a_1[a_1 \leftarrow a]\langle\!\langle 1/a \rangle\!\rangle \arrow[r, "{[\![\cdot]\!]_u}"] \arrow[d] & (\nu a)([\![a_1[a_1 \leftarrow a]]\!]_u \mid [\![1]\!]_a) \arrow[d] \\
\mathtt{fail}^{\emptyset} \arrow[r, "{[\![\cdot]\!]_u}"] & u.\overline{\mathtt{none}}
\end{tikzcd}
$$

FSCD'21-LMCS'23

- $\lambda^{\lightning}_{\oplus}$ and $\widehat{\lambda}^{\lightning}_{\oplus}$: subject reduction / subject expansion (for well-typed terms)
- Translation of $\lambda^{\lightning}_{\oplus}$ into $\widehat{\lambda}^{\lightning}_{\oplus}$ (variable atomization)
- Translation of intersection types (for $\widehat{\lambda}^{\lightning}_{\oplus}$) into session types (for $s\pi$)

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TYPES'21

- Extension of $\lambda_{\oplus}^{\lightning}$ and $\widehat{\lambda}_{\oplus}^{\lightning}$ with unrestricted resources
- Relies on client and servers in sπ (typable via $?A$ and $!A$, copying semantics)

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- A lazy semantics for commitment; translation of terms/types unchanged

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TYPES'21

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APLAS'23

- Variants of $\lambda^{\lightning}_{\oplus}$ and sπ with non-confluent non-determinism (linear resources)
- A lazy semantics for commitment; translation of terms/types unchanged

MFPS'24

- $\lambda^{\lightning}_{\oplus}$ and sπ with non-confluent non-determinism (with unrestricted resources)
- An eager semantics for commitment; lazy and eager translations compared

Concluding Remarks

- A glimpse at a series of concurrent interpretations (governed by session types) of resource λ-calculi with non-idempotent intersection types
- Reconciling non-determinism, failures, and confluence across functional and concurrent paradigms
- Focus on a basic setting with linear resources and confluent non-determinism. Extensions:
  - Unrestricted resources
  - Non-confluent non-determinism
  - Lazy/eager semantics for typed processes

Concluding Remarks

- A glimpse at a series of concurrent interpretations (governed by session types) of resource λ-calculi with non-idempotent intersection types
- Reconciling non-determinism, failures, and confluence across functional and concurrent paradigms
- Focus on a basic setting with linear resources and confluent non-determinism. Extensions:
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Current/future work:

- Recursive types
- Quantitative properties for typed process behaviors

Intersection Types Meet Session Types

Reconciling Two Different Views on Computational Resources

Jorge A. Pérez

University of Groningen, The Netherlands

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